

UNIFORMLY BOUNDED ORTHONORMAL SECTIONS OF POSITIVE LINE BUNDLES ON COMPLEX MANIFOLDS

BERNARD SHIFFMAN

To Duong H. Phong on the occasion of his 60th birthday

ABSTRACT. We show the existence of uniformly bounded sequences of increasing numbers of orthonormal sections of powers L^k of a positive holomorphic line bundle L on a compact Kähler manifold M . In particular, we construct for each positive integer k , orthonormal sections $s_1^k, \dots, s_{n_k}^k$ in $H^0(M, L^k)$, $n_k \geq \beta \dim H^0(M, L^k)$, such that $\{s_j^k\}$ is a uniformly bounded family, where β is an explicit positive constant depending only on the dimension of M . For $m = 1$, we can take $\beta = .99564$.

1. INTRODUCTION

In [Bo], Bourgain constructed a uniformly bounded orthonormal basis for the Hilbert space of holomorphic polynomials on the 3-sphere $S^3 \subset \mathbb{C}^2$. An open question is whether a bounded basis exists in higher dimensions, i.e., on $S^{2m+1} \subset \mathbb{C}^{m+1}$ for $m \geq 2$, or more generally on the boundary of a relatively compact strictly pseudoconvex domain D in a complex manifold Y . An important case, which in fact drives this research, is where Y is the dual bundle L^{-1} of a positive line bundle L over a compact Kähler manifold M , and $D \rightarrow M$ is the circle bundle consisting of elements of L^{-1} of length 1; we then seek a bounded orthonormal basis for the space $\mathcal{H}^2(D)$ of CR holomorphic functions on D . Specifically, we identify $\mathcal{H}^2(D)$ with the direct sum $\bigoplus_{k=0}^{\infty} H^0(M, L^k)$ of holomorphic sections of powers of L (see Section 2), and we conjecture that there exists a uniformly bounded sequence of orthonormal bases for the spaces $H^0(M, L^k)$, $k \in \mathbb{Z}^+$. Indeed, if $L \rightarrow M = \mathbb{CP}^m$ is the hyperplane section bundle, then $D = S^{2m+1}$, and the Hilbert space $\mathcal{H}^2(D)$ is the L^2 completion of the space of polynomials on \mathbb{C}^{m+1} restricted to S^{2m+1} . In this case, $H^0(\mathbb{CP}^m, L^k)$ is the space of homogeneous holomorphic polynomials of degree k on \mathbb{C}^{m+1} .

In this paper, we give a partial answer to the question of the existence of uniformly bounded orthonormal bases:

THEOREM 1.1. *Let $(L, h) \rightarrow (M, \omega)$ be a Hermitian holomorphic line bundle over a compact Kähler manifold, with positive curvature Θ_h and Kähler form $\omega = \frac{i}{2}\Theta_h$. Then there exist positive constants C, β such that for each positive integer k , we can find sets of orthonormal holomorphic sections*

$$s_1^k, \dots, s_{n_k}^k \in H^0(M, L^k), \quad n_k \geq \beta \dim H^0(M, L^k),$$

such that $\|s_j^k\|_{\infty} \leq C$ for $1 \leq j \leq n_k$, for all $k \in \mathbb{Z}^+$.

Date: April 5, 2014.

Research partially supported by NSF grant DMS-1201372.

The inner product on $H^0(M, L^k)$ is the L^2 inner product induced from the metric h^k on L^k and the volume form $\frac{1}{m!}\omega^m$ on M , where we let $m = \dim M$; see (3). The sup-norm is given by $\|s^k\|_\infty = \sup_{z \in M} |s^k(z)|_{h^k(z)}$, for $s^k \in H^0(M, L^k)$. We recall that

$$\dim H^0(M, L^k) = \frac{1}{m!} c_1(L)^m k^m + O(k^{m-1}), \quad (1)$$

by the Riemann-Roch Theorem and Kodaira Vanishing Theorem, so n_k grows at the rate k^m .

To compare Theorem 1.1 with known results, we note that the author and Zelditch showed in [SZ2] that for $2 < p < \infty$, random sections in $H^0(M, L^k)$ of unit L^2 norm satisfy a uniform L^p bound independent of k except for rare events of probability $< \exp(-Ck^{2m/p})$. Thus randomly chosen sequences of orthonormal bases will almost surely have uniform L^p bounds for all $p < \infty$. But random sections in $H^0(M, L^k)$ of unit L^2 norm will have L^∞ norms approximately equal to $\sqrt{m \log k}$ with high probability [FZ] (see also [SZ2]), so random sequences will almost surely not be uniformly bounded.

We shall also give explicit positive constants β_m depending only on the dimension m of M such that Theorem 1.1 holds for all $\beta < \beta_m$. (See Theorem 4.1.) For example, for $\dim M = 1$, there exist uniformly bounded orthonormal sections $s_1^k, \dots, s_{n_k}^k \in H^0(M, L^k)$ with

$$n_k \geq (.99564) \dim H^0(M, L^k), \quad \text{for all } k \in \mathbb{Z}^+.$$

Theorem 4.1 also gives upper bounds for $\limsup_{k \rightarrow \infty} [\max_{1 \leq j \leq n_k} \|s_j^k\|_\infty]$. The following result is a consequence of Theorem 4.1:

THEOREM 1.2. *Let $(L, h) \rightarrow (M, \omega)$ be as in Theorem 1.1. Then there exists a sequence of holomorphic sections $s^k \in H^0(M, L^k)$, $k = 1, 2, 3, \dots$, such that $\|s^k\|_2 = 1$ for all $k \in \mathbb{Z}^+$ and*

$$\limsup_{k \rightarrow \infty} \|s^k\|_\infty \leq \kappa_m \text{Vol}(M)^{-1/2},$$

where κ_m is a universal constant depending only on $m = \dim M$.

For a continuous section $\psi \in \mathcal{C}^0(M, L^k)$, one trivially has $\|\psi\|_\infty / \|\psi\|_2 \geq \text{Vol}(M)^{-1/2}$, with equality if and only if $|\psi|_{h^k}$ is constant. Thus, the constant κ_m can be regarded as a measure of the asymptotic “flatness” of the sections $\{s^k\}$.

Applying Theorem 1.2 to the case where $M = \mathbb{CP}^m$ and L is the hyperplane section bundle $\mathcal{O}(1)$ with the Fubini-Study metric, so that $H^0(\mathbb{CP}^m, L^k)$ can be identified with the space of homogeneous holomorphic polynomials of degree k on \mathbb{C}^{m+1} , with the L^p norm of a section given by the L^p norm of the corresponding polynomial over the unit sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$, we obtain the following result:

COROLLARY 1.3. *For all $m \geq 1$, there exists a sequence of homogeneous holomorphic polynomials p_k on \mathbb{C}^{m+1} such that $\deg p_k = k$ and*

$$\sup_k \frac{\|p_k\|_{L^\infty(S^{2m+1})}}{\|p_k\|_{L^2(S^{2m+1})}} < +\infty.$$

The result of Bourgain [Bo] mentioned at the beginning of this paper gives uniformly bounded sequences of orthonormal bases for the spaces $H^0(\mathbb{CP}^1, L^k)$ (which we identify with

the spaces of holomorphic homogeneous polynomials on \mathbb{C}^2). These bases are of the form

$$s_j^k = \frac{1}{\sqrt{k+1}} \sum_{q=0}^k e^{2\pi i j q / (k+1)} \sigma_q \chi_q^k, \quad 1 \leq j \leq k+1, \quad (2)$$

where $\chi_0^k, \chi_1^k, \dots, \chi_k^k$ are the L^2 normalized monomials in $H^0(\mathbb{CP}^1, L^k)$, and $\sigma_q = \pm 1$. (A deep part of the argument in [Bo] is to choose the signs of the σ_q appropriately to obtain uniform bounds.) Our method is to begin with coherent states peaked at “lattice points” in the manifold M in place of the monomials χ_q^k which peak along circles in \mathbb{CP}^1 . While the monomials are orthogonal, the coherent states are only approximately orthogonal. So we then modify the coherent states to make them orthogonal before constructing our orthonormal sections s_j^k as oscillating sums of the form (2) (but without the σ_q).

Corollary 1.3 implies the following result on spherical harmonics:

COROLLARY 1.4. *Let $m \geq 1$, and let Δ denote the Laplacian on the round sphere S^{2m+1} . Then there exists a sequence of (real) eigenfunctions f_k such that $\Delta f_k = \lambda_k f_k$, where $\lambda_k = k(k+2m)$ denotes the k -th eigenvalue of Δ , and*

$$\sup_k \frac{\|f_k\|_{L^\infty(S^{2m+1})}}{\|f_k\|_{L^2(S^{2m+1})}} < +\infty.$$

Proof (assuming Corollary 1.3). Let $p_k = u_k + iv_k$ be as in Corollary 1.3. Then u_k and v_k are (real) eigenfunctions of the Laplacian on the sphere with eigenvalue λ_k . We can choose p_k such that $\|u_k\|_2 \geq \|v_k\|_2$ and thus $\|u_k\|_2 \geq \|p_k\|/\sqrt{2}$. We then let $f_k = u_k$. \square

There are related open problems concerning the growth of L^∞ norms of eigenfunctions on compact Riemannian manifolds. For example, as far as we are aware, it remains unknown if there are uniformly bounded sequences $\{f_k\}$ of eigenfunctions satisfying $\Delta f_k = \lambda_k f_k$ with unit L^2 norms on even-dimensional round spheres. VanderKam [Va] showed that random sequences of eigenfunctions f_k of unit L^2 norm on S^2 satisfy the growth condition $\|f_k\|_\infty = O((\log k)^2)$. We note that Toth and Zelditch [TZ] observed that the only compact Riemannian manifolds with completely integrable geodesic flow and with eigenvalues of bounded multiplicity and which carry a uniformly bounded orthonormal basis of eigenfunctions are flat. A condition for manifolds to have less than maximal eigenfunction growth is given in [SoZ].

The author would like to thank Zhiqin Lu, Dror Varolin, Steve Zelditch, and Junyan Zhu for useful suggestions.

2. BACKGROUND

We review in this section background on geometry and the Szegő kernel from [BSZ, SZ1, SZ3, SZZ].

We let (L, h) be a positive Hermitian holomorphic line bundle over a compact complex manifold M of dimension m , as in Theorem 1.1. If we let e_L denote a nonvanishing local holomorphic section over an open set $\Omega \subset M$, then the curvature form of (L, h) is given locally over Ω by

$$\Theta_h = -\partial\bar{\partial} \log |e_L|_h^2.$$

Positivity of (L, h) means that the curvature Θ_h is positive, so that $\omega := \frac{i}{2}\Theta_h$ is a Kähler form on M .

The Hermitian metric h on L induces Hermitian metrics h^k on the powers L^k of the line bundle, and we give the space $H^0(M, L^k)$ of global holomorphic sections of L^k the Hermitian inner product

$$\langle s_k, s'_k \rangle = \int_M h^k(s_k, \overline{s'_k}) \frac{1}{m!} \omega^m, \quad s_k, s'_k \in H^0(M, L^k), \quad (3)$$

induced by the metrics h, ω . As in [BSZ, SZ1, SZZ], we lift sections $s_k \in H^0(M, L^k)$ to the circle bundle $X \xrightarrow{\pi} M$ of unit vectors in the dual bundle $L^{-1} \rightarrow M$ endowed with the dual metric h^{-1} . Since (L, h) is positive, X is a strictly pseudoconvex CR manifold. The lift $\hat{s}_k : X \rightarrow \mathbb{C}$ of the section s_k is given by

$$\hat{s}_k(\lambda) = (\lambda^{\otimes k}, s_k(z)) \quad , \quad \lambda \in \pi^{-1}(z) .$$

The sections \hat{s}_k span the space $\mathcal{H}_k^2(X)$ of CR holomorphic functions \hat{s} on X satisfying $\hat{s}(e^{i\theta}x) = e^{ik\theta}\hat{s}(x)$. This provides isomorphisms

$$H^0(M, L^k) \approx \mathcal{H}_k^2(X), \quad s_k \mapsto \hat{s}_k .$$

We shall henceforth identify $H^0(M, L^k)$ with $\mathcal{H}_k^2(X)$ by identifying s_k with \hat{s}_k .

The *Szegő projector* of level k is the orthogonal projector $\Pi_k : L^2(X) \rightarrow \mathcal{H}_k^2(X)$, which is given by the *Szegő kernel*

$$\Pi_k(x, y) = \sum_{j=1}^{d_k} \hat{S}_j^k(x) \overline{\hat{S}_j^k(y)} \quad (x, y \in X) ,$$

where $\{\hat{S}_1^k, \dots, \hat{S}_{d_k}^k\}$ is an orthonormal basis of $\mathcal{H}_k^2(X)$ and $d_k = \dim \mathcal{H}_k^2(X)$. It was shown in [Ca, Ti, Ze] (see also [BBS]) that the Szegő kernel on the diagonal has the asymptotics:

$$\Pi_k(x, x) = \frac{k^m}{\pi^m} + O(k^{m-1}) . \quad (4)$$

We write

$$P_k(z, w) := \frac{|\Pi_k(x, y)|}{\sqrt{\Pi_k(x, x)} \sqrt{\Pi_k(y, y)}}, \quad x \in \pi^{-1}(z), \quad y \in \pi^{-1}(w) . \quad (5)$$

We shall apply the following off-diagonal asymptotics of this normalized Szegő kernel:

PROPOSITION 2.1. *Let $b, q \in \mathbb{R}^+$. Then*

$$P_k(z, w) = \begin{cases} e^{-\frac{k}{2}[1+o(1)]\text{dist}(z, w)^2}, & \text{uniformly for } \text{dist}(z, w) \leq b \sqrt{\frac{\log k}{k}} \\ O(k^{-q}), & \text{uniformly for } \text{dist}(z, w) \geq \sqrt{(2q + 2m + 1) \frac{\log k}{k}} \end{cases} .$$

The estimates in Proposition 2.1 follow from Propositions 2.6–2.8 of [SZ3]. (See also [BBS, MM]. In fact, one has the bound $P_k(z, w) = O(e^{-C\sqrt{k}\text{dist}(z, w)})$ [Ch, De, Li], which is sharper than the above when $\text{dist}(z, w) > k^{-1/2+\varepsilon}$, and the first estimate of the proposition holds for $\text{dist}(z, w) < k^{-1/3}$ [SZ1], but the estimates of the proposition suffice for our purposes.)

We shall apply Proposition 2.1 with $q = m + 1$.

3. PROOF OF THEOREM 1.1

The uniformly bounded sections we construct are the opposite of peak sections which maximize the sup norm. We can think of these bounded sections as “flat sections” since they lack large peaks. They will be constructed as linear combinations of peak sections centered at “lattice points” in the following steps:

- (1) Construct peak sections at lattice-like points; these sections will be approximately orthonormal;
- (2) modify the sections to be orthonormal;
- (3) construct a family of linear combinations of these modified sections so they will be orthonormal and uniformly bounded.

3.1. Step 1: approximately orthonormal peak sections. Here we follow the method in [SZZ] based on the Szegő kernel asymptotics of Proposition 2.1. We repeat the argument here, since we need slightly sharper estimates.

Choose a point $z_0 \in M$ and identify $T_{z_0}M$ with \mathbb{R}^{2m} . Let

$$C_t := [-t, t]^{2m} \subset \mathbb{R}^{2m} \equiv T_{z_0}M$$

denote the $2m$ -cube of width $2t$ centered at the origin. Let $\gamma > 1$ be arbitrary, and choose t sufficiently small so that

$$\gamma^{-1}\|v - w\| \leq \text{dist}(\exp_{z_0}(v), \exp_{z_0}(w)) \leq \gamma\|v - w\|, \quad \text{for } v, w \in C_{2t}. \quad (6)$$

For each $k > 0$, we construct a lattice of points $\{z_\nu^k\}$ in M as follows: Let

$$\Gamma_k = \left\{ (\nu_1, \dots, \nu_{2m}) \in \mathbb{Z}^{2m} : |\nu_j| \leq \frac{t\sqrt{k}}{a} \right\}, \quad (7)$$

where a is to be chosen later. The number n_k of points in Γ_k is given by

$$n_k = \left(2 \left\lfloor \frac{t\sqrt{k}}{a} \right\rfloor + 1 \right)^{2m} = \left(\frac{2t}{a} \right)^{2m} k^m + O(k^{m-1/2}). \quad (8)$$

It follows from (8) and the Riemann-Roch theorem (1) that $n_k \geq \beta \dim H^0(M, L^k)$ for some positive constant β .

We now begin our construction of n_k orthonormal sections in $H^0(M, L^k) \equiv \mathcal{H}_k^2(X)$: We let

$$z_\mu^k = \exp_{z_0} \left(\frac{a}{\sqrt{k}} \mu \right) \in \exp_{z_0}(C_t), \quad \text{for } \mu \in \Gamma_k. \quad (9)$$

We choose points $y_\mu^k \in X$ with $\pi(y_\mu) = z_\mu$, where we omit the superscript k to simplify notation. We consider the L^2 -normalized *coherent states*

$$\Phi_\mu^k(x) := \frac{\Pi_k(x, y_\mu)}{\sqrt{\Pi_k(y_\mu, y_\mu)}} \quad (\mu \in \Gamma_k), \quad (10)$$

at the lattice points z_μ . These sections are “almost” orthonormal, since

$$\langle \Phi_\mu^k, \Phi_\nu^k \rangle = \frac{\int \Pi_k(x, y_\mu) \Pi_k(y_\nu, x) dx}{\sqrt{\Pi_k(y_\mu, y_\mu)} \sqrt{\Pi_k(y_\nu, y_\nu)}} = \frac{\Pi_k(y_\nu, y_\mu)}{\sqrt{\Pi_k(y_\mu, y_\mu)} \sqrt{\Pi_k(y_\nu, y_\nu)}}, \quad (11)$$

and thus

$$|\langle \Phi_\mu^k, \Phi_\nu^k \rangle| = P_k(z_\mu, z_\nu), \quad (12)$$

which decays rapidly for $\mu \neq \nu$, thanks to Proposition 2.1.

3.2. Step 2: orthonormal peak sections. The next step is to modify the set $\{\Phi_\nu^k\}$ of coherent states to obtain an orthonormal set $\{\Psi_\nu^k\}$. As in [SZZ], we consider the Hermitian $n_k \times n_k$ matrices

$$\Delta_k = (\Delta_{\mu\nu}), \quad \Delta_{\mu\nu} = \langle \Phi_\mu^k, \Phi_\nu^k \rangle, \quad \text{for } \mu, \nu \in \Gamma_k.$$

We note that by (11), the diagonal entries of Δ_k are 1. Since $|\Delta_{\mu\nu}| = P_k(z_\mu, z_\nu)$, Proposition 2.1 (with $q = m + 1$) says that

$$|\Delta_{\mu\nu}| \leq e^{[-1+o(1)] \frac{k}{2} \text{dist}(z_\mu, z_\nu)^2}, \quad \text{if } \text{dist}(z_\mu, z_\nu) \leq b\sqrt{\frac{\log k}{k}}, \quad (13)$$

$$|\Delta_{\mu\nu}| = O(k^{-m-1}), \quad \text{if } \text{dist}(z_\mu, z_\nu) \geq b\sqrt{\frac{\log k}{k}}, \quad (14)$$

where $b = \sqrt{4m+3}$.

It was shown in [SZZ, p. 1987] that for all $\eta > 0$, we can choose the constant a in (9) such that

$$\max_{\mu \in \Gamma_k} \left(\sum_{\nu \in \Gamma_k \setminus \{\mu\}} |\Delta_{\mu\nu}| \right) \leq \eta, \quad \text{for } k \gg 0. \quad (15)$$

We give below a simplified proof of (15), which yields an estimate for a :

Fix an element $\mu_0 \in \Gamma_k$. By (8) and (14), we have

$$\sum_{\nu \in \Gamma_k \setminus \{\mu_0\}} |\Delta_{\mu_0\nu}| = \sum_{\nu \in \Gamma_k(\mu_0)} |\Delta_{\mu_0\nu}| + O(k^{-1}),$$

where

$$\Gamma_k(\mu_0) = \left\{ \nu \in \Gamma_k : 0 < \text{dist}(z_{\mu_0}, z_\nu) \leq b\sqrt{\frac{\log k}{k}} \right\}.$$

Let $\varepsilon > 0$ be arbitrary. By (13),

$$\sum_{\nu \in \Gamma_k(\mu_0)} |\Delta_{\mu_0\nu}| \leq \sum_{\nu \in \Gamma_k(\mu_0)} e^{-\frac{k}{2}(1-\varepsilon) \text{dist}(z_{\mu_0}, z_\nu)^2}, \quad \text{for } k \gg 0.$$

Now let $a' = a/\gamma$, and let $\tilde{a} = a'\sqrt{1-\varepsilon}$. Since $\text{dist}(z_\nu, z_{\mu_0}) > \frac{a'}{\sqrt{k}}\|\nu - \mu_0\|$, we then have by (13)

$$\begin{aligned} \sum_{\nu \in \Gamma_k(\mu_0)} |\Delta_{\mu_0\nu}| &\leq \sum_{\nu \in \Gamma_k(\mu_0)} e^{-\tilde{a}^2 \|\nu - \mu_0\|^2/2} = \sum_{\nu \in \mathbb{Z}^m \setminus 0} e^{-\tilde{a}^2 \|\nu\|^2/2} = \left[\sum_{j=-\infty}^{\infty} e^{-\tilde{a}^2 j^2/2} \right]^{2m} - 1 \\ &< \left[1 + \int_{-\infty}^{\infty} e^{-\tilde{a}^2 x^2/2} dx \right]^{2m} - 1 = \left(1 + \tilde{a}^{-1} \sqrt{2\pi} \right)^{2m} - 1, \end{aligned}$$

for $k \gg 0$. Thus (15) holds whenever

$$\left(1 + \tilde{a}^{-1} \sqrt{2\pi} \right)^{2m} \leq 1 + \eta. \quad (16)$$

Recall that the $\ell^\infty \rightarrow \ell^\infty$ mapping norm of a linear map $A \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ is given by

$$\|A\|_{\ell^\infty \rightarrow \ell^\infty} = \sup\{\|Av\|_{\ell^\infty} : v \in \mathbb{C}^n, \|v\|_{\ell^\infty} = 1\} = \max_{1 \leq \mu \leq n} \sum_{\nu=1}^n |A_{\mu\nu}|, \quad (17)$$

where $\|v\|_{\ell^\infty} = \max_{1 \leq \mu \leq n} |v_\mu|$. By (15)–(17), we have:

LEMMA 3.1. *Let $\Delta_k = I - A_k$. Suppose that*

$$a > \frac{\gamma \sqrt{2\pi}}{(1 + \eta)^{1/2m} - 1}, \quad (18)$$

where γ satisfies the distortion bound (6). Then

$$\limsup_{k \rightarrow \infty} \|A_k\|_{\ell^\infty \rightarrow \ell^\infty} < \eta. \quad (19)$$

In the following, we let $0 < \eta < 1$, and we let a satisfy (18), so that $\|A_k\|_{\ell^\infty \rightarrow \ell^\infty} \leq \eta < 1$ for $k \gg 0$. It follows that the eigenvalues of Δ_k are bounded below by $1 - \eta$, and therefore Δ_k is invertible for $k \gg 0$. From the Taylor series

$$(1 - x)^{-1/2} = \sum_{j=0}^{\infty} \frac{(2j)!}{4^j j!^2} x^j,$$

it follows that the (positive definite Hermitian) square root of Δ_k^{-1} is given by

$$\Delta_k^{-1/2} = I + \sum_{j=1}^{\infty} \frac{(2j)!}{4^j j!^2} A_k^j, \quad (20)$$

where the series converges in the $\ell^\infty \rightarrow \ell^\infty$ mapping norm, for $k \gg 0$. Furthermore by (19),

$$\begin{aligned} \|\Delta_k^{-1/2}\|_{\ell^\infty \rightarrow \ell^\infty} &\leq 1 + \sum_{j=1}^{\infty} \frac{(2j)!}{4^j j!^2} \|A_k^j\|_{\ell^\infty \rightarrow \ell^\infty} \leq \sum_{j=0}^{\infty} \frac{(2j)!}{4^j j!^2} \|A_k\|_{\ell^\infty \rightarrow \ell^\infty}^j \\ &= (1 - \|A_k\|_{\ell^\infty \rightarrow \ell^\infty})^{-1/2} \leq (1 - \eta)^{-1/2}, \quad \text{for } k \gg 0. \end{aligned} \quad (21)$$

We write $\Delta_k^{-1/2} = B_k = (B_{\mu\nu})$. To complete Step 2, we define the “quasi-coherent states”

$$\Psi_\mu^k = \sum_{\nu \in \Gamma_k} B_{\mu\nu} \Phi_\nu^k \in H^0(M, L^k), \quad \text{for } \mu \in \Gamma_k. \quad (22)$$

By definition,

$$\langle \Psi_\mu^k, \Psi_\nu^k \rangle = \sum_{\rho, \sigma \in \Gamma_k} B_{\mu\rho} \bar{B}_{\nu\sigma} \Delta_{\rho\sigma} = \delta_\mu^\nu,$$

and thus the Ψ_μ^k are orthonormal.

3.3. Step 3: orthonormal flat sections. Our orthonormal uniformly bounded (“flat”) sections $\{s_j^k\}$ are easily constructed from our orthonormal peak sections Ψ_μ^k : Let $\zeta = e^{2\pi i/n_k}$ be a primitive (n_k) -th root of unity, and let $\tau^1, \tau^2, \dots, \tau^{n_k}$ be the lexicographic (or any other) ordering of the elements of Γ_k . We let

$$s_j^k = \frac{1}{\sqrt{n_k}} \sum_{q=1}^{n_k} \zeta^{qj} \Psi_{\tau^q}^k, \quad 1 \leq j \leq n_k. \quad (23)$$

Since the $\Psi_{\tau^q}^k$ are orthonormal,

$$\langle s_j^k, s_l^k \rangle = \frac{1}{n_k} \sum_{q=1}^{n_k} \zeta^{(j-l)q} = \delta_l^j, \quad \text{for } 1 \leq j, l \leq n_k,$$

and thus $\{s_j^k\}$ is an orthonormal family. To verify that the s_j^k are uniformly bounded, we consider the linear maps

$$F_k : \mathbb{C}^{n_k} \rightarrow \mathcal{H}_k^2(X), \quad (v_1, \dots, v_{n_k}) \mapsto \sum_{j=1}^{n_k} v_j \Phi_{\tau^j}^k,$$

with the $\ell^\infty \rightarrow L^\infty(X)$ mapping norm

$$\|F_k\|_{\ell^\infty \rightarrow L^\infty(X)} = \sup \left\{ \sup_{x \in X} \left| \sum_{\mu \in \Gamma_k} v_\mu \Phi_\mu^k(x) \right| : |v_\mu| \leq 1, \text{ for } \mu \in \Gamma_k \right\} = \sup_{x \in X} \sum_{\mu \in \Gamma_k} |\Phi_\mu^k(x)|.$$

LEMMA 3.2. $\|F_k\|_{\ell^\infty \rightarrow L^\infty(X)} \leq c k^{m/2}$, for some constant $c < +\infty$.

Proof. Let $x \in X$ be arbitrary, and let $z = \pi(x) \in M$. Then by (4),

$$\Phi_\mu^k(x) = \Pi_k(x, x)^{1/2} P_k(z, z_\mu) = (\pi^{-m/2} + o(1)) k^{m/2} P_k(z, z_\mu). \quad (24)$$

We consider two cases:

Case 1: $z \notin \exp_{z_0}(C_{2t})$. Then $d(z, z_\mu) \geq \text{dist}(z, C_t) \geq t/\gamma$ for all μ and k , so by (14), $P_k(z, z_\mu) = O(k^{-m-1})$ and hence by (8) and (24), $\sum_\mu |\Phi_\mu^k(x)| = o(k^{m/2})$ uniformly for $x \in X \setminus \pi^{-1}(\exp_{z_0}(C_{2t}))$.

Case 2: $z = \exp_{z_0}(p)$, where $p \in C_{2t}$. As before, we suppose that $\varepsilon > 0$ and we let $a' = a/\gamma$, $\tilde{a} = a'\sqrt{1-\varepsilon}$. As in the proof of (15),

$$\sum_{\mu \in \Gamma_k} P_k(z, z_\mu) \leq \sum \left\{ e^{-\frac{k}{2}(1-\varepsilon)\text{dist}(z, z_\mu)^2} : \text{dist}(z, z_\mu) \leq b\sqrt{\frac{\log k}{k}} \right\} + O(k^{-1}).$$

By (6),

$$\text{dist}(z, z_\mu) \geq \gamma^{-1} \left\| p - \frac{a}{\sqrt{k}} \mu \right\| = \frac{a'}{\sqrt{k}} \|p^k - \mu\|, \quad p^k = \frac{\sqrt{k}}{a} p.$$

Therefore

$$\begin{aligned} \sum_{\mu \in \Gamma_k} e^{-\frac{k}{2}(1-\varepsilon)\text{dist}(z, z_\mu)^2} &\leq \sum_{\mu \in \mathbb{Z}^{2m}} e^{-\tilde{a}^2 \|\mu - p^k\|^2/2} \leq \sum_{\mu \in \mathbb{Z}^{2m}} e^{-\tilde{a}^2 \|\mu\|^2/2} \\ &< \left[1 + \int_{-\infty}^{\infty} e^{-\tilde{a}^2 x^2/2} dx \right]^{2m} = \left(1 + \tilde{a}^{-1} \sqrt{2\pi} \right)^{2m}, \end{aligned}$$

where the second inequality is by the Poisson summation formula applied to the function $f(x) = e^{-\tilde{a}^2 x^2/2}$. The estimate of the lemma then follows from (24). \square

Completion of the proof of Theorem 1.1. It remains to show that the s_j^k are uniformly bounded. Combining (22)–(23), we have

$$s_j^k = \frac{1}{\sqrt{n_k}} \sum_{q=1}^{n_k} \sum_{\nu \in \Gamma_k} \zeta^{qj} B_{\tau^q \nu} \Phi_\nu^k.$$

Fix j and let

$$v_\nu = \sum_{q=1}^{n_k} \zeta^{qj} B_{\tau^q \nu}, \quad \text{for } \nu \in \Gamma_k.$$

By Lemma 3.1 and (21),

$$|v_\nu| \leq \sum_{q=1}^{n_k} |B_{\tau^q \nu}| = \sum_{q=1}^{n_k} |B_{\nu \tau^q}| = \|\Delta_k^{-1/2}\|_{\ell^\infty \rightarrow \ell^\infty} \leq (1 - \eta)^{-1/2}, \quad (25)$$

where $\eta = (1 + \tilde{a}^{-1} \sqrt{2\pi})^m - 1$. Thus by Lemma 3.2 and (25),

$$\|s_j^k\|_{L^\infty(X)} = \frac{1}{\sqrt{n_k}} \left\| \sum_{\nu \in \Gamma_k} \Phi_\nu^k v_\nu \right\|_{L^\infty(X)} \leq \frac{1}{\sqrt{n_k}} \|F_k\|_{\ell^\infty \rightarrow L^\infty(X)} \|B_k\|_{\ell^\infty \rightarrow \ell^\infty} \leq \frac{c}{\sqrt{1-\eta}} \frac{k^{m/2}}{\sqrt{n_k}},$$

for k sufficiently large. By (8), $n_k \geq \beta' k^m$ for a positive constant β' , and therefore the s_j^k are uniformly bounded above. \square

4. UNIVERSAL BOUNDS

In this section, we modify the above argument to obtain a universal value (depending only on the dimension of M) of the fraction β in Theorem 1.1. We also give a universal bound for the asymptotic sup norms of the orthonormal sections:

THEOREM 4.1. *There exist positive constants β_m depending only on $m \in \mathbb{Z}^+$ such that if $(L, h) \rightarrow (M, \omega)$ is as in Theorem 1.1, with $\dim M = m$, then for all $\beta < \beta_m$, there exist sets of orthonormal holomorphic sections*

$$s_1^k, \dots, s_{n_k}^k \in H^0(M, L^k), \quad n_k \geq \beta \dim H^0(M, L^k),$$

such that the family $\{s_j^k : 1 \leq j \leq n_k, k \in \mathbb{Z}^+\}$ is uniformly bounded.

Furthermore, there exist constants $\kappa_m(\beta)$, depending only on m and β , such that the s_j^k can be chosen to satisfy the universal asymptotic L^∞ bound

$$\limsup_{k \rightarrow \infty} \left[\max_{1 \leq j \leq n_k} \|s_j^k\|_\infty \right] \leq \kappa_m(\beta) \text{Vol}(M)^{-1/2}. \quad (26)$$

Proof. Let $a_m, \beta_m \in \mathbb{R}^+$ be given by

$$\sum_{j=-\infty}^{+\infty} e^{-a_m^2 j^2/2} = 2^{1/2m}, \quad \beta_m = \pi^m / a_m^{2m}. \quad (27)$$

Suppose that $\beta < \beta_m$, and choose $a > a_m$ such that $\beta < \pi^m / a^{2m}$. Then choose $\tilde{a} > a_m$ and $\gamma > 1$ such that $a_m < \tilde{a} < a/\gamma$. We decompose M into a finite number of disjoint domains

$\{U_j\}_{1 \leq j \leq q}$ with piecewise smooth boundaries such that $M = \bigcup_{j=1}^q \overline{U_j}$ and that there exist points $p_j \in U_j$ and open sets $W_j \subset T_{p_j} M \equiv \mathbb{R}^{2m}$ such that $\exp_{p_j}(W_j) = U_j$,

$$\gamma^{-1} \|v - w\| \leq \text{dist}(\exp_{p_j}(v), \exp_{p_j}(w)) \leq \gamma \|v - w\|, \quad \text{for } v, w \in W_j, \quad (28)$$

and writing $\exp_{p_j}^*(\frac{1}{m!}\omega^m) = g_{(j)} dx_1 \wedge \cdots \wedge dx_{2m}$,

$$0 < g_{(j)}(v) \leq 1 + \frac{\delta}{\text{Vol}(M)}, \quad \text{for } v \in W_j, \quad (29)$$

where $\delta > 0$ is to be chosen later.

Choose smooth domains $U_j'' \subset\subset U_j' \subset\subset U_j$ such that $\text{Vol}(M \setminus \bigcup U_j'') < \delta$, and let $W_j'' = \exp_{p_j}^{-1}(U_j'')$. By (29),

$$\text{Vol}(U_j'') = \int_{W_j''} g_{(j)} dx_1 \wedge \cdots \wedge dx_{2m} \leq \left(1 + \frac{\delta}{\text{Vol}(M)}\right) \text{Vol}(W_j''),$$

and therefore

$$\sum_{j=1}^q \text{Vol}(W_j'') \geq \left(1 - \frac{\delta}{\text{Vol}(M)}\right) \sum_{j=1}^q \text{Vol}(U_j'') \geq \text{Vol}(M) - 2\delta. \quad (30)$$

For $k \in \mathbb{Z}^+$, we write

$$z_{\mu j}^k = \exp_{p_j} \left(\frac{a}{\sqrt{k}} \mu \right) \quad \text{for } \mu \in \mathbb{Z}^{2m}, 1 \leq j \leq q, \quad (31)$$

and we let

$$\Gamma_{kj} = \left\{ \mu \in \mathbb{Z}^{2m} : \frac{a}{\sqrt{k}} \mu \in W_j'' \right\} = \{ \mu \in \mathbb{Z}^{2m} : z_{\mu j}^k \in U_j'' \}.$$

We thus obtain a collection of “lattice points” $\{z_{\mu j}^k : \mu \in \Gamma_{kj}, 1 \leq j \leq q\}$ throughout M . As before, we choose points $y_{\mu j}^k \in X$ with $\pi(y_{\mu j}^k) = z_{\mu j}^k$, and we consider the family of coherent states

$$\Phi_{\mu j}^k(x) := \frac{\Pi_k(x, y_{\mu j}^k)}{\Pi_k(y_{\mu j}^k, y_{\mu j}^k)^{1/2}} \in \mathcal{H}_k^2(X), \quad \mu \in \Gamma_{kj}, j = 1, \dots, q,$$

at these lattice points. It follows from (30) that the number n_k of lattice points satisfies the inequality

$$n_k = \sum_{j=1}^q \#(\Gamma_{kj}) = \frac{k^m}{a^{2m}} \sum_{j=1}^q \text{Vol}(W_j'') + O(k^{m-1/2}) > \frac{\text{Vol}(M) - 3\delta}{a^{2m}} k^m, \quad \text{for } k \gg 0. \quad (32)$$

To show that n_k satisfies the lower bound of the theorem, we recall that the volume of M is given by

$$\text{Vol}(M) = \int_M \frac{1}{m!} \omega^m = \frac{1}{m!} \int_M [\pi c_1(L, h)]^m = \frac{\pi^m}{m!} c_1(L)^m. \quad (33)$$

Let $\varepsilon = \pi^m/a^{2m} - \beta > 0$. Then by (1) and (32),

$$\frac{n_k}{\dim H^0(M, L^k)} > \frac{m! [\text{Vol}(M) - 3\delta]}{c_1(L)^m a^{2m}} = \frac{\pi^m - 3m! \delta / c_1(L)^m}{a^{2m}} = \beta + \varepsilon - \frac{3m!}{a^{2m} c_1(L)^m} \delta,$$

for $k \gg 0$. Choosing $\delta \leq \frac{1}{3m!} a^{2m} c_1(L)^m \varepsilon$, we obtain the desired bound

$$n_k > \beta \dim H^0(M, L^k). \quad (34)$$

We now construct n_k orthonormal sections of $\mathcal{H}_k^2(X) = H^0(M, L^k)$ for k sufficiently large. Following the approach of Section 3.2, we define the Hermitian $n_k \times n_k$ matrices

$$\Delta_k = (\Delta_{(\mu j)(\nu l)}), \quad \Delta_{(\mu j)(\nu l)} = \langle \Phi_{\mu j}^k, \Phi_{\nu l}^k \rangle, \quad \text{for } \mu \in \Gamma_{kj}, \nu \in \Gamma_{kl}, 1 \leq j, l \leq q.$$

Recalling that $\tilde{a} > a_m$, we let

$$\eta = \left(\sum_{j=-\infty}^{+\infty} e^{-\tilde{a}^2 j^2 / 2} \right)^{2m} - 1 < 1.$$

Fix μ_0, j_0 , with $\mu_0 \in \Gamma_{kj_0}$. Since $\tilde{a} < a/\gamma$, we see by the argument in Section 3.2 that

$$\sum_{(\mu, j) \neq (\mu_0, j_0)} \Delta_{(\mu_0 j_0)(\mu j)} \leq \sum_{\nu \in \mathbb{Z}^m \setminus 0} e^{-\tilde{a}^2 \|\nu\|^2 / 2} + O(k^{-1}) = \eta + O(k^{-1}).$$

Therefore, for k sufficiently large, Δ_k is invertible and we have

$$\|\Delta_k^{-1/2}\|_{\ell^\infty \rightarrow \ell^\infty} \leq (1 - \eta)^{-1/2} + O(k^{-1}).$$

We can then construct as before the orthonormal family

$$\Psi_{\mu j}^k = \sum_{\nu l} [\Delta_k^{-1/2}]_{(\mu j)(\nu l)} \Phi_{\nu l}^k \in \mathcal{H}_k^2(X), \quad \text{for } k \gg 0. \quad (35)$$

We let $\tau^1, \tau^2, \dots, \tau^{n_k}$ be an (arbitrary) ordering of the indices (μj) . As in Section 3.3, we define the orthonormal sections

$$s_j^k = \frac{1}{\sqrt{n_k}} \sum_{q=1}^{n_k} \zeta^{qj} \Psi_{\tau^q}^k, \quad 1 \leq j \leq n_k. \quad (36)$$

and we consider the linear maps

$$F_k : \mathbb{C}^{n_k} \rightarrow \mathcal{H}_k^2(X), \quad (v_1, \dots, v_n) \mapsto \sum_{j=1}^{n_k} v_j \Phi_{\tau^j}^k.$$

By the proof of Lemma 3.2 (with C_{2t} replaced with $\bigcup U_j'$), we conclude that

$$\|F_k\|_{\ell^\infty \rightarrow L^\infty(X)} \leq (1 + \eta + o(1)) \frac{k^{m/2}}{\pi^{m/2}}.$$

for k sufficiently large. It then follows as before that

$$\|s_j^k\|_{L^\infty(X)} \leq \frac{1}{\sqrt{n_k}} \|F_k\|_{\ell^\infty \rightarrow L^\infty(X)} \|\Delta_k^{-1/2}\|_{\ell^\infty \rightarrow \ell^\infty} \leq \frac{1 + \eta + o(1)}{\pi^{m/2} \sqrt{1 - \eta}} \frac{k^{m/2}}{\sqrt{n_k}},$$

for k sufficiently large. Since $\dim H^0(M, L^k) = \frac{1}{\pi^m} \text{Vol}(M) [k^m + o(k^{m-1})]$, it follows from (34) that

$$n_k > \frac{\beta}{\pi^m} \text{Vol}(M) k^m, \quad \text{for } k \gg 0,$$

and therefore

$$\|s_j^k\|_{L^\infty(X)} < \frac{1 + \eta}{\sqrt{\beta(1 - \eta)}} \text{Vol}(M)^{-1/2}, \quad \text{for } k \gg 0. \quad (37)$$

□

Remark: The bound in (37) depends on the choice of $\tilde{a} < a/\gamma < a < \pi^{1/2}/\beta^{1/2m}$. However, if one chooses a sequence $\tilde{a}(\nu) \nearrow \pi^{1/2}/\beta^{1/2m}$, then from the resulting sequences $\{s_j^k(\nu)\}$ we can construct $\{s_j^k\}$ satisfying (37) with

$$\eta = \left[\sum_{j=-\infty}^{+\infty} \exp\left(-\frac{\pi j^2}{2\beta^{1/m}}\right) \right]^{2m} - 1.$$

4.1. Numerical values for β_m . Solving (27) numerically using *Maple 18*, we obtain the following values (to 5 decimal places): $\beta_1 \approx .99220$, $\beta_2 \approx .44342$, $\beta_3 \approx .17782$, $\beta_4 \approx .06630$, $\beta_5 \approx .02345$, $\beta_6 \approx .00796$.

However, one obtains larger values of the constants β_m in Theorem 4.1 by using the lattice points

$$z_{\mu j}^k = \exp_{p_j} \left[\frac{a}{\sqrt{k}} (\mu_1 + e^{i\pi/3}\mu_2, \mu_3 + e^{i\pi/3}\mu_4, \dots, \mu_{2m-1} + e^{i\pi/3}\mu_{2m}) \right], \quad (38)$$

for $\mu = (\mu_1, \dots, \mu_{2m}) \in \mathbb{Z}^{2m}$ (where we identify $T_{p_j}M \equiv R^{2m} \equiv \mathbb{C}^m$), instead of the points of the lattice (31). In place of (32), we have

$$n_k > \left(\frac{2}{\sqrt{3}} \right)^m \frac{\text{Vol}(M) - 3\delta}{a^{2m}} k^m, \quad \text{for } k \gg 0.$$

We note that $|\mu_1 + e^{i\pi/3}\mu_2|^2 = \mu_1^2 + \mu_2^2 + \mu_1\mu_2$; then in place of (27), we let $\alpha_m, \beta'_m \in \mathbb{R}^+$ be given by

$$\sum_{\mu \in \mathbb{Z}^2} e^{-\alpha_m^2(\mu_1^2 + \mu_2^2 + \mu_1\mu_2)/2} = 2^{1/m}, \quad \beta'_m = \left(\frac{2}{\sqrt{3}} \right)^m \frac{\pi^m}{\alpha_m^{2m}} = \left(\frac{2\pi}{\sqrt{3}\alpha_m^2} \right)^m. \quad (39)$$

Let $\beta < \beta'_m$, and choose $\gamma, \alpha, \tilde{\alpha}$ such that $\alpha_m < \tilde{\alpha} < \alpha/\gamma$ and $\beta < \left(\frac{2\pi}{\sqrt{3}\alpha^2} \right)^m$. Repeating the proof of Theorem 4.1, we obtain the estimate

$$\sum_{(\mu, j) \neq (\mu_0, j_0)} \Delta_{(\mu_0 j_0)(\mu j)} \leq \left[\sum_{\mu \in \mathbb{Z}^2} e^{-\tilde{\alpha}^2(\mu_1^2 + \mu_2^2 + \mu_1\mu_2)/2} \right]^m - 1 + O(k^{-1}).$$

We then conclude that

$$\frac{n_k}{\dim H^0(M, L^k)} > \left(\frac{2}{\sqrt{3}} \right)^m \frac{\text{Vol}(M) - 3\delta}{c_1(L)^m \alpha^2} = \left(\frac{2}{\sqrt{3}} \right)^m \frac{\pi^m - 3\delta/c_1(L)^m}{\alpha^{2m}} > \beta,$$

for $k \gg 0$, if δ is chosen small enough. Thus Theorem 1.1 holds for all β less than the value of β'_m given by (39). Solving (39) numerically using *Maple 18*, one obtains the better values (to 5 decimal places): $\beta'_1 \approx .99564$, $\beta'_2 \approx .45867$, $\beta'_3 \approx .19254$, $\beta'_4 \approx .07572$, $\beta'_5 \approx .02838$, $\beta'_6 \approx .01024$. In dimension 1, the lattice (38) appears to give the best value of β_1 (compared with other lattices), but this lattice is probably not optimal for $m \geq 2$. An open question is whether the result can be improved further by using coherent states at other collections of points, e.g., Fekete points.

REFERENCES

- [BBS] R. Berman, B. Berndtsson and J. Sjöstrand, A direct approach to Bergman kernel asymptotics for positive line bundles, *Ark. Mat.* 46 (2008), 197–217.
- [BSZ] P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds, *Invent. Math.* 142 (2000), 351–395.
- [Bo] J. Bourgain, Applications of the spaces of homogeneous polynomials to some problems on the ball algebra. *Proc. Amer. Math. Soc.* 93 (1985), 277–283.
- [Ca] D. Catlin, The Bergman kernel and a theorem of Tian, in: *Analysis and Geometry in Several Complex Variables*, G. Komatsu and M. Kuranishi, eds., Birkhäuser, Boston, 1999.
- [Ch] M. Christ, Slow off-diagonal decay for Szegő kernels associated to smooth Hermitian line bundles, in: *Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001)*, *Contemp. Math.* 320, Amer. Math. Soc., Providence, RI, 2003, pp. 77–89.
- [De] H. Delin, Pointwise estimates for the weighted Bergman projection kernel in \mathbf{C}^n , using a weighted L^2 estimate for the $\bar{\partial}$ equation, *Ann. Inst. Fourier (Grenoble)* 48 (1998), 967–997.
- [FZ] R. Feng and S. Zelditch, Median and mean of the supremum of L^2 normalized random holomorphic fields, *J. Funct. Anal.*, to appear, arXiv:1303.4096.
- [Li] N. Lindholm, Sampling in weighted L^p spaces of entire functions in \mathbf{C}^n and estimates of the Bergman kernel, *J. Funct. Anal.* 182 (2001), 390–426.
- [MM] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, *Progress in Mathematics*, 254, Birkhäuser Verlag, Basel, 2007.
- [SZ1] B. Shiffman and S. Zelditch, Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds, *J. Reine Angew. Math.* 544 (2002), 181–222.
- [SZ2] B. Shiffman and S. Zelditch, Random polynomials of high degree and Levy concentration of measure, *Asian J. Math.* 7 (2003), 627–646.
- [SZ3] B. Shiffman and S. Zelditch, Number variance of random zeros on complex manifolds, *Geom. Funct. Anal.* 18 (2008), 1422–1475.
- [SZZ] B. Shiffman, S. Zelditch and S. Zrebiec, Overcrowding and hole probabilities for random zeros on complex manifolds, *Indiana Univ. Math. J.* 57 (2008), 1977–1997.
- [SoZ] C. Sogge and S. Zelditch, Riemannian manifolds with maximal eigenfunction growth, *Duke Math. J.* 114 (2002), 387–437.
- [Ti] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, *J. Diff. Geometry* 32 (1990), 99–130.
- [TZ] J. Toth and S. Zelditch, Riemannian manifolds with uniformly bounded eigenfunctions, *Duke Math. J.* 111 (2002), 97–132.
- [Va] J. VanderKam, L^∞ norms and quantum ergodicity on the sphere, *Internat. Math. Res. Notices* 1997 (1997), 329–347; Correction to “ L^∞ norms and quantum ergodicity on the sphere,” *Internat. Math. Res. Notices*, 1998 (1998), 65.
- [Ze] S. Zelditch, Szegő kernels and a theorem of Tian, *Internat. Math. Res. Notices* 1998 (1998), 317–331.

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA
 E-mail address: shiffman@math.jhu.edu